GEOMETRY AND TOPOLOGY PRELIMINARY EXAM, FALL 2016

Answer all six questions.

1. On $\mathbf{R}^3$, consider the one-form $\alpha = dz - ydx$. Let us define a distribution $D \subset T\mathbf{R}^3$ by $D = \ker(\alpha)$.
   (a) Give a frame for D (vector fields in D which span the fiber at all points) and determine whether D is integrable or not.
   (b) Give a curve whose tangent bundle lies in D and remark on why this is not a contradiction to your previous result.

2. Let $X = \mathbf{R}/\mathbf{Z}$ and define a cover $\{U, V\}$, where U is the image of $\{x \in \mathbf{R} \mid -\frac{1}{10} < x < \frac{6}{10}\}$ under the quotient map $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$. and similarly define V by $\{x \in \mathbf{R} \mid \frac{4}{10} < x < \frac{11}{10}$. Let ρ_U, ρ_V be a partition of unity.

   (a) Write down the associated short exact sequence of co-chain complexes and long exact sequence in (de Rham) cohomology.

   (b) Compute the image of $1 \in H^0_{dR}(U \cap V)$ under the coboundary map. Be clear about your construction.

   (c) Using the fact that $\dim H^k_{dR}(\mathbf{R}) = \begin{cases} 1 & k = 0, \\ 0, & k > 0, \end{cases}$ compute the cohomology groups of X.

3. Consider the unique simplicial complex structure, defined on a square-shaped closed planar domain, determined by taking the zero-simplicies to be the four corners together with the point at the center, so five total.
   (a) Show how to define a genus-one surface S by identifying the boundary of the square in an appropriate way.
   (b) In the induced Δ-complex structure on S defined by the simplicial complex structure above, give (two) explicit cochain representatives of a basis for $H_1(S, \mathbf{Z})$.
   (c) Show that the cup product of your two generators from Part (3b) generates $H_2(S, \mathbf{Z})$.

4. For this question, you may make use of any standard results, so long as they are referenced explicitly.
   (a) Identify the real projective space $\mathbb{RP}^2$ as a quotient of an n-gon.
   (b) From your answer to (1), compute the fundamental group
   $$\pi_1 \mathbb{RP}^2$$
   and prove your answer by applying the Seifert–van Kampen theorem.
   (c) Calculate the Euler characteristic $\chi(\mathbb{RP}^2)$.
   (d) Enumerate those finite groups G which act freely on $\mathbb{RP}^2$, and prove your answer.

5.
   (a) Calculate the dimension of the orthogonal group $\mathsf{O}(n)$ as a smooth manifold. Prove your answer.
   (b) Calculate the Euler characteristic $\chi\big(\mathsf{O}(n)\big)$. Prove your answer.
   (c) Prove that there is a diffeomorphism
   $$\mathsf{O}(3) \cong \mathbb{Z}/2 \times \mathbb{RP}^3$$
   between the orthogonal group and the disjoint union of two real projective spaces.

6. (a) State Sard's theorem.
   (b) Prove that if M is compact smooth manifold with boundary ∂M, there does *not* exist a smooth map $g : M \to \partial M$ which restricts to the identity on ∂M. You may make use of any standard results in differential topology in your proof.